Proof of trigonometric formulas

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1 Fundamentals

1.1 Trigonometric circle

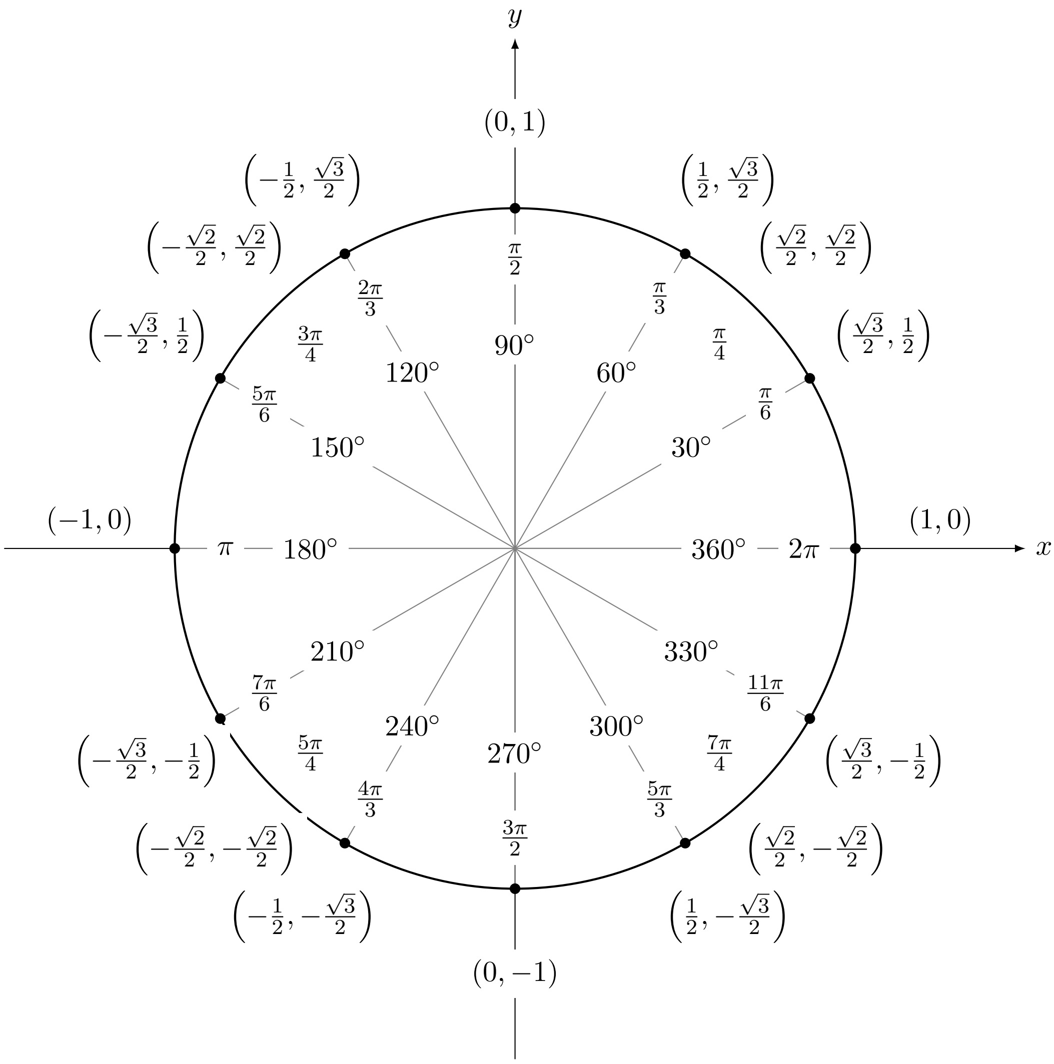

Figure 1: The unit circle with cosine and sine values for some common angles.

1.2 Basic relations

According to the trigonometric circle (cf. Figure 1) and the theorem of Pythagore :

$$\cos(x)^2 + \sin(x)^2 = 1 \tag{1}$$

Here are the fundamental relations between the trigonometric functions :

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \tag{2}$$

$$\cot(x) = \frac{1}{\tan(x)}$$

$$\begin{aligned}\csc(x) &= \frac{1}{\sin(x)} \implies \csc(x)^2 = \frac{\cos(x)^2+\sin(x)^2}{\sin(x)^2} = \cot(x)^2+1\\ \sec(x) &= \frac{1}{\cos(x)} \implies \sec(x)^2 = \frac{\cos(x)^2+\sin(x)^2}{\cos(x)^2} = \tan(x)^2+1\end{aligned}$$

1.3 Periodic properties

$$\begin{cases}\sin(x+2k\pi) = \sin(x)\\ \cos(x+2k\pi) = \cos(x)\\ \tan(x+k\pi) = \tan(x)\\ \cot(x+k\pi) = \cot(x)\end{cases} \qquad \begin{cases}\sin(-x) = -\sin(x)\\ \cos(-x) = \cos(x)\\ \tan(-x) = -\tan(x)\\ \cot(-x) = -\cot(x)\end{cases}$$

2 Duplication

2.1 Proof of duplication formulas

Let (a,b) $\in \mathbb{R}^2$:

$$\begin{cases} e^{ia} &= \cos(a) + i \cdot \sin(a) \\ e^{ib} &= \cos(b) + i \cdot \sin(b) \end{cases}$$

$$\begin{aligned} e^{i(a+b)} &= e^{ia} \times e^{ib} \\ &= (\cos(a) + i \cdot \sin(a)) \times (\cos(b) + i \cdot \sin(b)) \\ &= \cos(a)\cos(b) + i \cdot \cos(a)\sin(b) + i \cdot \sin(a)\cos(b) + i^2 \cdot \sin(a)\sin(b) \\ \cos(a+b) + i \cdot \sin(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) + i \cdot (\sin(a)\cos(b) + \cos(a)\sin(b)) \end{aligned}$$

So, in $\mathbb{R}$ and $\mathbb{C}$ we get the formulas :

$$\boxed{\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)} \quad (3)$$

$$\boxed{\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)} \quad (4)$$

According to the relations 2, 3 and 4, we get :

$$\begin{aligned} \tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} \\ \tan(a+b) &= \frac{\sin(a)\cos(b) + \cos(a)\sin(b)}{\cos(a)\cos(b) - \sin(a)\sin(b)} = \frac{\frac{\sin(a)\cos(b) + \cos(a)\sin(b)}{\cos(a)\cos(b)}}{\frac{\cos(a)\cos(b) - \sin(a)\sin(b)}{\cos(a)\cos(b)}} \end{aligned}$$

We finally get :

$$\boxed{\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}} \quad (5)$$

2.2 Proof of periodic identities

$$\begin{aligned} \sin\left(\frac{\pi}{2} - x\right) &= \sin\left(\frac{\pi}{2}\right)\cos(x) - \cos\left(\frac{\pi}{2}\right)\sin(x) \\ &= 1 \cdot \cos(x) - 0 \cdot \sin(x) \\ &= \cos(x) \end{aligned}$$

$$\begin{aligned} \cos\left(\frac{\pi}{2} - x\right) &= \cos\left(\frac{\pi}{2}\right)\cos(x) + \sin\left(\frac{\pi}{2}\right)\sin(x) \\ &= 0 \cdot \cos(x) + 1 \cdot \sin(x) \\ &= \sin(x) \end{aligned}$$

$$\begin{aligned}\tan\left(\frac{\pi}{2}-x\right) &= \tan\left(\frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)}\right)\\ &= \cot(x)\\ \cot\left(\frac{\pi}{2}-x\right) &= \cot\left(\frac{\cos(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)}\right)\\ &= \tan(x)\end{aligned}$$

$$\begin{aligned}\sin(\pi+x) &= \sin(\pi)\cos(x)+\cos(\pi)\sin(x)\\ &= 0\cos(x)+-1\sin(x)\\ &= -\sin(x)\\ \cos(\pi+x) &= \cos(\pi)\cos(x)-\sin(\pi)\sin(x)\\ &= -1\cos(x)-0\sin(x)\\ &= -\cos(x)\\ \tan(\pi+x) &= \frac{\sin(\pi+x)}{\cos(\pi+x)}\\ &= \tan(x)\\ \cot(\pi+x) &= \frac{\cos(\pi+x)}{\sin(\pi+x)}\\ &= \cot(x)\end{aligned}$$

$$\begin{aligned}\sin(\pi-x) &= \sin(\pi)\cos(x)-\cos(\pi)\sin(x)\\ &= 0\cos(x)-(-1)\sin(x)\\ &= \sin(x)\\ \cos(\pi-x) &= \cos(\pi)\cos(x)+\sin(\pi)\sin(x)\\ &= -1\cos(x)+0\sin(x)\\ &= -\cos(x)\\ \tan(\pi-x) &= \frac{\sin(\pi-x)}{\cos(\pi-x)}\\ &= -\tan(x)\\ \cot(\pi-x) &= \frac{\cos(\pi-x)}{\sin(\pi-x)}\\ &= -\cot(x)\end{aligned}$$

$$
\begin{aligned}
\sin(2x) &= \sin(x+x) \\
&= \sin(x)\cos(x) + \cos(x)\sin(x) \\
&= 2\sin(x)\cos(x) \\
&= \frac{2\sin(x)\cos(x)^2}{\cos(x)} \\
&= 2\tan(x)\cos(x)^2 \\
&= \frac{2\tan(x)}{\frac{1}{\cos(x)^2}} \textit{ (Calculation trick)} \\
&= \frac{2\tan(x)}{\frac{\sin(x)^2+\cos(x)^2}{\cos(x)^2}}
\end{aligned}
$$

$$
\boxed{\sin(2x) = \frac{2\tan(x)}{\tan(x)^2+1}} \tag{6}
$$

$$
\begin{aligned}
\cos(2x) &= \cos(x+x) \\
&= \cos(x)\cos(x) - \sin(x)\sin(x) \\
&= \cos(x)^2 - \sin(x)^2 \\
&= \frac{\frac{\cos(x)^2-\sin(x)^2}{\cos(x)^2}}{\frac{1}{\cos(x)^2}} \textit{ (Calculation trick)} \\
&= \frac{\frac{\cos(x)^2-\sin(x)^2}{\cos(x)^2}}{\frac{\sin(x)^2+\cos(x)^2}{\cos(x)^2}}
\end{aligned}
$$

$$
\boxed{\cos(2x) = \frac{1-\tan(x)^2}{\tan(x)^2+1}} \tag{7}
$$

$$
\begin{aligned}
\tan(2x) &= \frac{2\sin(x)\cos(x)}{\cos(x)^2-\sin(x)^2} \\
&= \frac{\frac{2\sin(x)\cos(x)}{\cos(x)^2}}{\frac{\cos(x)^2-\sin(x)^2}{\cos(x)^2}} \textit{ (Calculation trick)}
\end{aligned}
$$

$$
\boxed{\tan(2x) = \frac{2\tan(x)}{1-\tan(x)^2}} \tag{8}
$$

$$
\cos(2x) = 1 - 2\sin(x)^2
$$

$$\Leftrightarrow \boxed{\sin(x)^2 = \frac{1}{2}\Big(1 - \cos(2x)\Big)} \tag{9}$$

$$\Leftrightarrow \ \cos(2x) = 1 - 2(1 - \cos(x)^2) \ \textit{(cf. equation n°1)}$$

$$\Leftrightarrow \boxed{\cos(x)^2 = \frac{1}{2}\Big(1 + \cos(2x)\Big)} \tag{10}$$

$$\implies \boxed{\tan(x)^2 = \frac{1 - \cos(2x)}{1 + \cos(2x)}} \tag{11}$$

$$\begin{aligned}\cos(a-b) - \cos(a+b) &= \cos(a)\cos(b) + \sin(a)\sin(b) - (\cos(a)\cos(b) - \sin(a)\sin(b))\\ &= 2\sin(a)\sin(b)\end{aligned}$$

$$\Leftrightarrow \boxed{\sin(a)\sin(b) = \frac{1}{2}\Big(\cos(a-b) - \cos(a+b)\Big)} \tag{12}$$

$$\begin{aligned}\cos(a-b) + \cos(a+b) &= \cos(a)\cos(b) + \sin(a)\sin(b) + \cos(a)\cos(b) - \sin(a)\sin(b)\\ &= 2\cos(a)\cos(b)\end{aligned}$$

$$\Leftrightarrow \boxed{\cos(a)\cos(b) = \frac{1}{2}\Big(\cos(a-b) + \cos(a+b)\Big)} \tag{13}$$

$$\sin(a-b) + \sin(a+b) = \sin(a)\cos(b) - \cos(a)\sin(b) + \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\Leftrightarrow \boxed{\sin(a)\cos(b) = \frac{1}{2}\Big(\sin(a-b) + \sin(a+b)\Big)} \tag{14}$$

3 Addition

Let (a,b) $\in \mathbb{R}^2$:

$$\left\{\begin{array}{lll} a & = & x+y \\ b & = & x-y \end{array}\right. \Leftrightarrow \left\{\begin{array}{lll} x & = & \dfrac{a+b}{2} \\ y & = & \dfrac{a-b}{2} \end{array}\right.$$

$$\begin{aligned} \sin(a)+\sin(b) &= \sin(x+y)+\sin(x-y) \\ &= \sin(x)\cos(y)+\cos(x)\sin(y)+\sin(x)\cos(y)-\cos(x)\sin(y) \\ &= 2\sin(x)\cos(y) \end{aligned}$$

$$\boxed{\sin(a)+\sin(b) = 2\sin\left(\frac{a+b}{2}\right)\times\cos\left(\frac{a-b}{2}\right)} \tag{15}$$

$$\begin{aligned} \sin(a)-\sin(b) &= \sin(x+y)-\sin(x-y) \\ &= \sin(x)\cos(y)+\cos(x)\sin(y)-(\sin(x)\cos(y)-\cos(x)\sin(y)) \\ &= 2\cos(x)\sin(y) \end{aligned}$$

$$\boxed{\sin(a)-\sin(b) = 2\cos\left(\frac{a+b}{2}\right)\times\sin\left(\frac{a-b}{2}\right)} \tag{16}$$

$$\begin{aligned} \cos(a)+\cos(b) &= \cos(x+y)+\cos(x-y) \\ &= \cos(x)\cos(y)-\sin(x)\sin(y)+\cos(x)\cos(y)+\sin(x)\sin(y) \\ &= 2\cos(x)\cos(y) \end{aligned}$$

$$\boxed{\cos(a)+\cos(b) = 2\cos\left(\frac{a+b}{2}\right)\times\cos\left(\frac{a-b}{2}\right)} \tag{17}$$

$$\begin{aligned} \cos(a)-\cos(b) &= \cos(x+y)-\cos(x-y) \\ &= \cos(x)\cos(y)-\sin(x)\sin(y)-(\cos(x)\cos(y)+\sin(x)\sin(y)) \\ &= -2\sin(x)\sin(y) \end{aligned}$$

$$\boxed{\cos(a)-\cos(b) = -2\sin\left(\frac{a+b}{2}\right)\times\sin\left(\frac{a-b}{2}\right)} \tag{18}$$

4 Equation

To solve trigonometric equations we need to know some properties about the trigonometric functions. Here are the more usefull ones :

$$\forall\ (x,\alpha) \in \mathbb{R}^2 \text{ and } k \in \mathbb{Z}$$

$$\begin{aligned} \sin(x) = \sin(\alpha) &\Leftrightarrow x = \alpha + 2k\pi \\ &\Leftrightarrow x = \pi - \alpha + 2k\pi \end{aligned}$$

$$\begin{aligned} \cos(x) = \cos(\alpha) &\Leftrightarrow x = \alpha + 2k\pi \\ &\Leftrightarrow x = -\alpha + 2k\pi \end{aligned}$$

$$\tan(x) = \tan(\alpha) \Leftrightarrow x = \alpha + k\pi$$

5 Table of Formulas

Formula	Reference	Page
$\cos(x)^2 + \sin(x)^2 = 1$	1	2
$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$	2	2
$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$	3	4
$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$	4	4
$\tan(a \pm b) = \dfrac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$	5	4
$\sin(2x) = \dfrac{2\tan(x)}{\tan(x)^2 + 1}$	6	6
$\cos(2x) = \dfrac{1 - \tan(x)^2}{\tan(x)^2 + 1}$	7	6
$\tan(2x) = \dfrac{2\tan(x)}{1 - \tan(x)^2}$	8	6
$\sin(x)^2 = \dfrac{1}{2}\Big(1 - \cos(2x)\Big)$	9	7
$\cos(x)^2 = \dfrac{1}{2}\Big(1 + \cos(2x)\Big)$	10	7
$\tan(x)^2 = \dfrac{1 - \cos(2x)}{1 + \cos(2x)}$	11	7
$\sin(a)\sin(b) = \dfrac{1}{2}\Big(\cos(a-b) - \cos(a+b)\Big)$	12	7
$\cos(a)\cos(b) = \dfrac{1}{2}\Big(\cos(a-b) + \cos(a+b)\Big)$	13	7
$\sin(a)\cos(b) = \dfrac{1}{2}\Big(\sin(a-b) + \sin(a+b)\Big)$	14	7
$\sin(a) + \sin(b) = 2\sin\left(\dfrac{a+b}{2}\right) \times \cos\left(\dfrac{a-b}{2}\right)$	15	8
$\sin(a) - \sin(b) = 2\cos\left(\dfrac{a+b}{2}\right) \times \sin\left(\dfrac{a-b}{2}\right)$	16	8
$\cos(a) + \cos(b) = 2\cos\left(\dfrac{a+b}{2}\right) \times \cos\left(\dfrac{a-b}{2}\right)$	17	4
$\cos(a) - \cos(b) = -2\sin\left(\dfrac{a+b}{2}\right) \times \sin\left(\dfrac{a-b}{2}\right)$	18	8